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INVESTIGATION OF THE PROBLEM OF PLANE STRATIFIED FLOW AROUND A BODY†

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The problem of plane arbitrarily stratified flow of finite thickness around a body is considered in the linearized formulation. A theorem of Tikhonov and Samarskii [1] on the uniqueness of the solution of the external Dirichlet problem for the Helmholtz equation in the plane case is extended to asymmetrical radiation conditions and this is used to investigate the nature of the non-uniqueness of the solution of the linearized problem of the flow around the body. The existence of a solution is investigated by potential-theory methods. The "dipole approximations" of the solution of the flow problem when the radiation coefficient approaches zero are proved.

1. FORMULATION OF THE PROBLEM

THE PLANE steady-state flow of an ideal heavy incompressible liquid with an arbitrary stable stratification and a free boundary flowing around a body situated inside the flow or on a horizontal bottom is considered. The origin of coordinates is chosen to be on the bottom, the y axis is directed upwards, and the x axis is directed along the flow. As $x\to-\infty$ the flow is asymptotically unperturbed, its velocity is U, its depth is H, the characteristic dimension of the obstacle along the vertical is h, and the acceleration due to gravity is g. The quantities U and H are taken as the units of velocity and length, and the equations are written in dimensionless variables. The piecewise-smooth function $\rho(y)$ specifies the density distribution in the unperturbed flow, where $d\rho\leqslant 0$, $\rho\geqslant\rho_0>0$.

We will assume that the total energy of a particle in the unperturbed flow is sufficient for the particle to be able to rise in the field of the force of gravity from the equilibrium level at a height $h/2$. In this case the obstacle cannot "block the flow" and the pattern of streamlines has qualitatively the same character as in the case of uniform flow around the body. There are only two critical points on the boundary of the body and only one streamline branches at these critical points. On any streamline the x coordinate increases monotonically from $-\infty$ to $+\infty$.

If we put

$$-\rho'(y)/\rho(y)=\varepsilon a^2(y),\quad \nu=gH/U^2,\quad \varepsilon\nu=k^2$$

and denote by $\zeta(x,y)$ the deviation along the vertical of a liquid particle from the equilibrium state, the exact equation of motion has the form

$$\Delta\varsigma+a^2(y-\varsigma)(k^2\varsigma+\varepsilon\partial\varsigma/\partial y-\tfrac{1}{2}\varepsilon(\nabla\varsigma)^2)=0 \tag{1.1}$$

Suppose $y=Y(x)$ is the equation of the unknown free boundary, $x=x_0(s)$, and $y=y_0(s)$,

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$0 \leqslant s \leqslant s_0$ is the equation of the boundary γ of the body Ω. Then the system of boundary conditions has the form

$$\varsigma(x,0)=0,\quad \varsigma(x,Y(x))=Y(x)-1,\quad \lim_{x\to-\infty}\varsigma(x,y)=0 \tag{1.2}$$

$$\partial\varsigma/\partial y-\nu\varsigma-\tfrac{1}{2}(\nabla\varsigma)^2=0 \text{ when } y=Y(x)$$
$$\varsigma(x_0(s),\ y_0(s))=y_0(s)-\psi_0 \tag{1.3}$$

The unknown function $Y(x)$ and the unknown constant ψ_0 must be determined when solving the problem.

2. LINEARIZATION

We will assume that the parameter ε is fairly small and that the deviation of the ordinate of a liquid particle from its unperturbed value is also small, i.e. the dimensions of the body are small compared with the depth of the flow. By linearizing Eq. (1.1), we obtain the Helmholtz equation with refractive index $k^2a^2(y)$

$$\frac{\partial^2\varsigma}{\partial x^2}+\frac{\partial^2\varsigma}{\partial y^2}+k^2a^2(y)\varsigma=0 \tag{2.1}$$

No limitations are imposed on the value of the parameter k. After linearizing the boundary conditions (1.2) we obtain

$$\varsigma(x,0)=0,\quad \frac{\partial\varsigma}{\partial y}(x,1)-\nu\varsigma(x,1)=0,\quad \lim_{x\to-\infty}\varsigma(x,y)=0 \tag{2.2}$$

Boundary condition (1.3) is not changed.

3. INTEGRAL REPRESENTATION OF THE SOLUTION

As we know, Green's function $G(x, y, \eta)$ satisfies the inhomogeneous Helmholtz equation with right-hand side $\delta(x)\delta(y-\eta)$ and boundary conditions (2.2).

Using the method of separation of variables it is easy to show that [2]

$$G(x,y,\eta)=\theta(x)\sum_{n=1}^{N}\kappa_n^{-1}\varphi_n(y)\varphi_n(\eta)\sin(\kappa_n x)+\sum_{n=1+N}^{\infty}(2\kappa_n)^{-1}\varphi_n(y)\varphi_n(\eta)e^{-|\kappa_n x|} \tag{3.1}$$

where $\theta(x)$ is the Heaviside function, which equals zero when $x<0$ and equals unity when $x\geqslant 0$, $\{\varphi_n(y)\}$ is the orthonormalized system of eigenfunctions of the corresponding Sturm–Liouville problem, $\kappa_1^2, \ldots, \kappa_N^2$ are positive eigenvalues, and $-\kappa_i^2$, $i>N$ are negative eigenvalues.

If we use the well-known asymptotic eigenvalues and the eigenfunctions as $n\to\infty$, it can be shown that

$$G(x,y,\eta)\sim(2\pi)^{-1}\ln\sqrt{x^2+(y-\eta)^2}\quad \text{as}\quad x\to 0,\ y\to\eta \tag{3.2}$$

If we apply Green's formula and use relation (3.2) and the fact that $\zeta(x, y)\to 0$ as $x\to-\infty$, and $G(x-\xi, x, \eta)\to 0$ as $\xi\to+\infty$, then for the solution which possesses the correct normal derivative [3] on the contour γ we obtain the integral representation

$$\varsigma(x,y) = -\int_{\gamma} G(x-\xi, y, \eta)\frac{\partial\varsigma}{\partial\eta}(\xi,\eta)ds \tag{3.3}$$

4. THEOREM OF THE UNIQUENESS OF THE SOLUTION OF THE EXTERNAL DIRICHLET PROBLEM

Suppose T_Ω is a strip $0 \leqslant y \leqslant 1$, from which the interior of the region Ω is removed. We will formulate the external Dirichlet problem as follows: it is required to obtain the function $\zeta(x, y)$ twice continuously differentiable inside the interior of T_Ω, continuous on $\overline{T}_\Omega$, and which satisfies Eq. (2.1) and the boundary conditions (2.2) inside the region T_Ω, and also satisfies the condition $\varsigma|_\gamma = f(s)$, where $f(s)$ is a known continuous function.

The theorem of uniqueness. Suppose Ω is a convex region, and $a(y)$ is a piecewise-analytic function, $k > 0$. The solution of the external Dirichlet problem with correct normal derivative on γ is unique.

It is sufficient to show that when $f(s) \equiv 0$ the solution $\zeta(x, y) \equiv 0$. Suppose the region Ω lies in the rectangle $|x| < a$, $0 < y < 1$. Substituting expansion (3.1) into (3.3) we obtain that when $x > a$, the following equation holds

$$\varsigma(x,y) = \sum_{n=1}^{N} \alpha_n \sin\kappa_n(x-\delta_n)\varphi_n(y) + \tfrac{1}{2}\sum_{n=N+1}^{\infty} \alpha_n e^{-\kappa_n x}\varphi_n(y) \tag{4.1}$$

The expressions for the numbers α_n and δ_n in terms of curvilinear integrals over the contour γ are not important for our further discussion.

Consider the system of functions

$$\psi_k(x,y) = \begin{cases} \cos\kappa_x(x-\delta_k)\varphi_k(y), & k = 1,\ldots,N \\ e^{\kappa_k x}\varphi_k(y), & k > N \end{cases} \tag{4.2}$$

which satisfy Eq. (2.1) and boundary conditions (2.2) when $y = 0$ and $y = 1$. Applying Green's formula to the functions $\zeta(x, y)$ and $\psi_k(x, y)$ in the rectangle $a \leqslant x \leqslant b$, $0 \leqslant y \leqslant 1$, we obtain

$$\int_C \left(\psi_k \frac{\partial\varsigma}{\partial n} - \varsigma\frac{\partial\psi_k}{\partial n}\right)ds = 0, \quad k \geqslant 1 \tag{4.3}$$

By virtue of the boundary conditions the integrals over the horizontal sides of the rectangle vanish. Substituting (4.1) and (4.2) into (4.3) and using the orthonormality of the system $\varphi_k(y)$, we obtain that the integrals over the vertical sides of the rectangle are independent of a and b and are equal to $\kappa_k a_k$. Consequently $\zeta \equiv 0$ when $x > a$. We can similarly prove that $\zeta \equiv 0$ when $x < -a$.

Suppose $\{0, y_1, \ldots, y_{n-1}, 1\}$ is a partition of the section $(0, 1)$ such that in each of the intervals (y_{i-1}, y_i) the function $a(y)$ is analytic, and T_i is the strip $-\infty < x < +\infty$, $0 < y < 1$, $T_{i\Omega} = T_i \setminus \Omega$. In view of the convexity of the region Ω, the domain $T_{i\Omega}$ is split into two connectivity components, and when $(x; y) \in T_{i\Omega}$ and $|x| > a$ the function $\zeta(x, y) \equiv 0$. Since the function $\zeta(x, y)$ is analytic in the region $T_{i\Omega}$, then by virtue of the theorem of uniqueness for analytic functions $\zeta(x, y) \equiv 0$ in $T_{i\Omega}$ $(i = 1, \ldots, n)$. Hence, $\zeta(x, y) \equiv 0$ in the region T_Ω.

If we make the stronger assumption that $a(y)$ is analytic in (0, 1), we can dispense with the condition of the convexity of the region Ω.

The flow problem will henceforth be investigated by potential-theory methods. The internal homogeneous Neumann problem will play an important role here, namely, it is required to obtain a function $\zeta(x, y)$ that is continuous in $\overline{\Omega}$ and twice continuously differentiable inside the region Ω and having a correct normal derivative on γ equal to zero.

Those values of the parameter k for which the homogeneous internal Neumann problem has non-trivial solutions, will be called critical values. We know that the critical values, which are eigenvalues of the spectral problem for Laplace's equation, form a denumerable set, non-negative and of finite multiplicity.

5. SOLUTION OF THE FLOW PROBLEM BY POTENTIAL-THEORY METHODS

It follows from (3.2) that the functions

$$V^{(0)} = -2\int_0^{s_0} \mu(s)G(x(s)-x, y(s), y, k)ds$$

$$V^{(1)} = -2\int_0^{s_0} \nu(s)\frac{\partial}{\partial n_{\xi\eta}}(x-x(s), y, y(s), k)ds$$

possess all the properties of the potentials of simple and double layers [3].

If we seek a solution of the external Dirichlet problem in the form of a double-layer potential, and a solution of the internal homogeneous Neumann problem in the form of a simple-layer potential, we obtain an associated Fredholm integral equation for determining the densities of the layers

$$\mu(\sigma) = 2\int_0^{s_0} \mu(s)K(s,\sigma,k)ds \tag{5.1}$$

$$\nu(\sigma) = 2\int_0^{s_0} \nu(s)K(\sigma,s,k)ds + f(\sigma) \tag{5.2}$$

$$K(\sigma,s,k) = \frac{\partial}{\partial n_{\xi\eta}} G(x(\sigma)-x(s),\ y(\sigma), y(s), k)$$

Those values of the parameter k for which Eq. (5.1) has non-trivial solutions will be called critical values.

Suppose k is a critical value, while $\mu_1(s), \ldots, \mu_m(s)$ are the corresponding eigenfunctions of Eq. (5.1). We know [3] that Eq. (5.2) has a solution if and only if the function $f(s)$ is orthogonal to $\mu_1(s), \ldots, \mu_m(s)$.

Note that $k=0$ is a simple critical value. The corresponding solution of the internal Neumann problem is identically equal to unity in the region Ω. If this solution is represented in the form of a simple-layer potential, the density of this potential will be denoted by $\mu_0(s)$. When $k=0$, Eq. (5.1) and the corresponding external Dirichlet problem for Laplace's equation have a solution if and only if the function $f(s)$ is orthogonal to $\mu_0(s)$.

If k is not a critical value, then, as follows from potential theory, Eq. (5.2) and the external Dirichlet problem are solvable for an arbitrary continuous function $f(s)$. It is interesting that when $k>0$, the external Dirichlet problem is solvable even if k is a critical value. This assertion was proved in [4] for symmetrical radiation conditions. We will present a proof for the plane case and for asymmetrical radiation conditions.

Suppose $k>0$ is a critical value of multiplicity m, and $\mu_1(s), \ldots, \mu_m(s)$ are the corresponding eigenfunctions of Eq. (5.1). We will seek a solution of the external Dirichlet problem in the following form

$$\varsigma(x,y) = V^{(1)}(x,y) + \sum_{i=1}^{m} \alpha_i G(x-a_i, y, b_i, k), \quad (a_i, b_i) \in \Omega \tag{5.3}$$

To determine the density $\nu(\sigma)$ we will obtain the integral equation

$$\nu(\sigma) = 2\int_0^{s_0} \nu(s)K(\sigma,s,k)ds + f(\sigma) + \sum_{i=1}^{m} \alpha_i G(x(\sigma) - a_i,\ y(\sigma), b_i, k)$$

For this equation to be solvable it is necessary and sufficient that the free term should be orthogonal to $\mu_1(\sigma), \ldots, \mu_m(\sigma)$. To determine the constants α we obtain the system of linear equations

$$\sum_{i=1}^{m} \alpha_i V_j(a_i, b_i) = c_j, \quad c_j = -\int_0^{s_0} f(\sigma)\mu_j(\sigma)d\sigma \tag{5.4}$$

$$V_j(x,y) = \int_0^{s_0} G(x(\sigma) - x, y, y(\sigma), k)\mu_j(\sigma)d\sigma$$

System (5.4) is solvable if and only if its determinant is non-zero

$$\det\left|V_j(a_i, b_i)\right| \neq 0 \tag{5.5}$$

We will show that we can choose points $(a_i,\ b_i)$ so that conditions (5.5) are satisfied. If this is not so, then for any points $(x_i,\ y_i) \in \Omega$, $i = 1, \ldots, m$ the following condition will be satisfied

$$F(x_1, y_1, \ldots, x_m, y_m) = \det\left|V_j(x_i, y_i)\right| = 0 \tag{5.6}$$

Since the simple-layer potential $V_j(x,\ y)$ is continuous in the closed region T, relation (5.6) is satisfied for any points $(x_i,\ y_i) \in \overline{\Omega}$. We now note that for each pair of variables the function F is a solution of the Helmholtz equation in the external region $T \backslash \Omega$, which vanishes at the boundary of the region. In view of the uniqueness of the solution of the external Dirichlet problem the relation $F \equiv 0$ is satisfied over the whole strip T. Hence, the jump in the normal derivative of the function F on passing through the boundary of the region is equal to zero for any group $(\xi_i,\ \eta_i)$. Using the rule for calculating the derivative of a determinant and the fact that the jump of the normal derivative of the simple-layer potential on passing through the boundary of the region is equal to the density of the potential from (5.6), we obtain that

$$\det\left|\mu_j(\sigma_i)\right| = 0 \tag{5.7}$$

for any set of points $\sigma_1, \ldots, \sigma_m$ in the region $[0,\ s_0]$. As can easily be shown, it follows from the linear independence of the functions $\mu_1(\sigma), \ldots, \mu_m(\sigma)$ that we obtain in the section $[0,\ s_0]$ those points $\sigma_1, \ldots, \sigma_m$ such that $\det|\mu_j(\sigma_i)| \neq 0$, which contradicts Eq. (5.7).

The contradiction obtained proves that a choice of points $(a_i,\ b_i) \in \Omega$ is possible for which condition (5.5) is satisfied and, consequently, system (5.4) and correspondingly the external Dirichlet problem for the Helmholtz equation have a solution.

Note that the solution is simplified when $m = 1$.

6. INVESTIGATION OF THE PROBLEM OF THE FLOW AROUND A BODY

When solving the flow problem we must take $y_0(s) - \psi_0$ as the function $f(s)$, as follows from the boundary condition (1.3), where ψ_0 is an arbitrary parameter in the section [0, 1]. It follows from the proved uniqueness and existence of the solution of the Dirichlet problem that the solution of the flow problem depends linearly on the arbitrary parameter ψ_0.

Using the model of an ideal liquid we can dispense with the non-uniqueness only by taking into account additional hypotheses. In the case of a homogeneous liquid we can use the hypothesis that the circulation of the velocity is equal to zero or the Zhukovskii–Chaplygin hypothesis that the velocity is finite at the sharp edge of a wing. If the liquid is inhomogeneous, then for an arbitrary value of k the question of a reasonable additional hypothesis remains open. Here we will adopt the approach applicable in the case of a slightly stratified liquid, i.e.

when $k \to 0$.

Since the critical values of k are isolated and $k=0$ is a simple critical value, we will take k in the range $[0,\ k_0]$ in which there are no other critical values apart from $k=0$. It has been shown that the problem of the flow around a plane region reduces to solving the external Dirichlet problem for the Helmholtz equation, where two parameters ψ_0 and k occur in the equations and boundary conditions. In view of the uniqueness of the solution of the external Dirichlet problem the solution of the flow problem will depend on the parameters ψ_0 and k.

If we seek a solution of the flow problem in the form

$$\varsigma = -2\int_0^{s_0} \nu(s)\frac{\partial G}{\partial n_{\xi\eta}}(x-x(s),y,y(s),k)ds + \alpha G(x-a,y,b) \tag{6.1}$$

then, to determine the density $\nu(\sigma)$ we obtain the following integral equation

$$\nu(\sigma,k) = 2\int_0^{s_0} K(\sigma,s,k)\nu(s,k)ds + \alpha G(x(\sigma)-a,y(\sigma),b,k) + y(\sigma) - \psi_0 \tag{6.2}$$

The function K is given by Eq. (5.2).

When $k>0$, the solution of Eq. (6.2) depends on five parameters: after substituting the solution of Eq. (6.2) into (6.1) the dependence on the parameters α, a and b should drop out, since ζ depends only on the parameters k and ψ_0. Hence, the parameters α, a and b can be chosen arbitrarily.

When $k=0$, Eq. (6.2) has a solution if and only if the following condition is satisfied

$$\alpha\int_0^{s_0} G_0(x(\sigma)-a,y(\sigma),b)\mu_0(\sigma)d\sigma + \int_0^{s_0}(y(\sigma)-\psi_0)\mu_0(\sigma)d\sigma = 0,\ G_0 = G|_{k=0} \tag{6.3}$$

where $\mu_0(\sigma)$ is the solution of the associated homogeneous equation.

It was shown earlier that one can always choose a and b so that the coefficient of the parameter α does not vanish. This enables us to determine α from Eq. (6.3).

Since when $k>0$ the choice of the parameter α is arbitrary, we will take it to be the same as when $k=0$. The solution of Eq. (6.2) will then depend continuously on the parameter k in the whole region $[0,\ k_0]$.

If the flow is non-circulatory and $\alpha=0$, Eq. (6.3) enables us to determine ψ_0, since the coefficient of this parameter cannot be zero. In fact, if this is not so, unity is orthogonal to $\mu_0(\sigma)$ and the external Dirichlet problem for Laplace's equation with the function $f(s)\equiv 1$ has a solution in the form of a double-layer potential

$$\varsigma = -2\int_0^{s_0} \nu(s)\frac{\partial G_0}{\partial n_{\xi\eta}}(x-x(s),y,y(s))ds \tag{6.4}$$

It follows from representation (6.4) and from the properties of the function G_0 that $\int_\gamma(\partial\zeta/\partial n)ds = 0$ for any closed contour γ surrounding the region Ω.

It follows from (3.1) that

$$G_0(x,y,\eta) = \theta(x)\kappa^{-1}\varphi(y)\varphi(\eta)\sin\kappa x + G_1(x,y,\eta) \tag{6.5}$$

where the function G_1 is even with respect to x and falls exponentially as $x\to\infty$. Substituting (6.5) into (6.4) we obtain that when $|x|>a$ and a is fairly large, the following relation holds

$$\varsigma(x,y) = C\theta(x)\varphi(y)\sin\kappa(x-x_0) + \varsigma_1(x,y)$$

where C and x_0 are certain numbers, while the function $\zeta_1(x,\ y)\to 0$ as $x\to\infty$. We will put $x_n = x_0 + \pi n/\kappa$ and $T_n = \{(x,\ y)\colon (x,\ y)\in T\setminus\Omega,\ |x|\le x_n\}$.

Applying Green's formula to the region T_n and using the boundary conditions and the choice of κ_n, we obtain

$$-\int\limits_{T_n}\int(\nabla\varsigma)^2 dxdy - \nu\int\limits_{-\infty}^{x_n}\varsigma^2(x,1)dx + \int\limits_{\gamma}\varsigma\frac{\partial\varsigma}{\partial n}ds \to 0 \quad \text{as} \quad n\to\infty \tag{6.6}$$

Since $\varsigma|_\gamma = 1$, the last integral in (6.6) is equal to zero, and consequently $\nabla\varsigma = 0$ in the region T_n. Since n is arbitrary, we have $\nabla\varsigma = 0$ in the region $T\setminus\Omega$ and $\zeta = \text{const}$ in $T\setminus\Omega$. But this is impossible since $\varsigma|_\gamma = 1$ and $\zeta \to 0$ as $x\to-\infty$. Thus, when $\alpha = 0$, Eq. (6.4) determines the value of ψ_0.

7. DIPOLE APPROXIMATIONS

We will write (6.1) in the form

$$\varsigma = -2\int\limits_{\gamma}\nu(s,k)\left(\frac{\partial G}{\partial\xi}(x-\xi,y,\eta,k)d\eta - \frac{\partial G}{\partial\eta}(x-\xi,y,\eta,k)d\xi\right)+ \tag{7.1}$$
$$+\alpha G(x-x_0,y,y_0,k)$$

If the dimensions of the body around which the flow occurs are small compared with the depth of the liquid, and the point $(x_0,\ y_0)\in\Omega$, then, apart from quantities of higher orders with respect to the dimensions of the body, we obtain from (7.1)

$$\varsigma(x,y) = -2A(k)\frac{\partial G}{\partial\xi}(x-x_0,y,y_0,k)+$$
$$+2B(k)\frac{\partial G}{\partial\eta}(x-x_0,y,y_0,k)+\alpha G(x-x_0,y,y_0,k) \tag{7.2}$$
$$A(k) = \int\limits_{\gamma}\nu(s,k)d\eta, \quad B(k) = \int\limits_{\gamma}\nu(s,k)d\xi, \quad x >> x_0$$

where the constant α is found from (6.3).

It is natural to call (7.2) the "dipole approximation" of the solution of the flow problem.

The coefficients $A(k)$ and $B(k)$ depend analytically on the parameter k, and for small k can be replaced by $A(0)$ and $B(0)$; the error resulting from this replacement is of the order of k^2.

We will show how the coefficients $A(0)$ and $B(0)$ can be obtained. We put $k=0$ in (7.1) and use the fact that when $k=0$ the function $\zeta = \text{Im}(z=w(z))$, where $w(z)$ is the complex potential and $G=\text{Im}H$, where $H(z,\ \varsigma,\ \bar\varsigma)$ is a known analytic function having a logarithmic singularity when $z=\zeta$

$$H(z,z_0,\bar z_0) = i(2\pi)^{-1}\ln(z-z_0)+H_1(z,z_0,\bar z_0) \tag{7.3}$$

It follows from (7.3) that

$$z-w(z) = -2A_0H_\xi + 2B_0H_\eta + \alpha H = \tag{7.4}$$
$$= -iA_0\pi^{-1}(z-z_0)^{-1} - B_0\pi^{-1}(z-z_0)^{-1} + H_2(z,z_0,\bar z_0)$$

where the function $H_2(z,\ z_0,\ \bar z_0)$ is regular at the point $z=z_0$.

It follows from (7.4) that $\alpha = \Gamma(2\pi)^{-1}$, where Γ is the circulation of the velocity around the contour γ. Differentiating (7.4), multiplying the result by $z-z_0$, and integrating over the contour γ taking into account the fact that $d\psi|_\gamma = 0$, we obtain

$$-\int_\gamma (z-z_0)d\varphi = 2A_0 - 2iB_0$$

$$2A_0 = -\int_\gamma (x-x_0)d\varphi,\quad 2B_0 = \int_\gamma (y-y_0)d\varphi$$

For arbitrary motion of the body with velocity (U, V) the complex potential can be represented in the form $w = Uw_1 + Vw_2 + \Gamma w_4$ [5], where $\psi_1|_\gamma = -y$, $\psi_2|_\gamma = -x$. In the case considered $w = -w_1 + \Gamma w_4 - z$, and hence

$$2B_0 = \int_\gamma (y-y_0)(-dx - d\varphi_1 + \Gamma d\varphi_4) = -\int_\gamma y dx - \int_\gamma \psi_1 d\varphi_1 +$$

$$+\int_\gamma (y-y_0)d\varphi_4 = -S - \lambda_{11} + \Gamma\eta_0$$

$$-2A_0 = \int_\gamma (x-x_0)(-dx - d\varphi_1 + \Gamma d\varphi_4) =$$

$$= \int_\gamma \psi_2 d\varphi_1 + \Gamma\int_\gamma (x-x_0)d\varphi_4 = -\lambda_{12} + \Gamma\xi_0$$

where S is the area of the region Ω, λ_{11} and λ_{12} are the corresponding additional masses, and (ξ_0, η_0) is the conformal centre of gravity of the region Ω [5].

Substituting the expressions for the coefficients into (7.2) we obtain that at distances that considerably exceed the diameter of the body, the following approximate formula holds

$$\varsigma(x,y) = (-\lambda_{12} + \Gamma\xi_0)G_\xi(x-x_0, y, y_0, k) +$$

$$+(-S - \lambda_{11} + \Gamma\eta_0)G_\eta(x-x_0, y, y_0, k) + \Gamma(2\pi)^{-1}G(x-x_0, y, y_0, k) \tag{7.5}$$

If $\Gamma = 0$ and $\lambda_{12} = 0$, we obtain a simpler formula from (7.5), obtained in [6] for the case of distributed sources.

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